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Ward Identities in CP^1 Nonlinear σ Model with Maxwell–Chern–Simons Term

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In $(2+1)$ space-time dimensions, CP^1 nonlinear σ model with Maxwell–Chern–Simons (MCS) term is studied by Ward identities. Firstly we revised, in the Coulomb gauge, the system is quantized in Faddeev–Senjanovic (FS) path integral quantization formalism. The canonical Ward identities are then given. Based on the Ward identities, the relations of the generating functional of proper vertex can be derived, and be expressed in Feynman rules with one-loop graphs.

KEY WORDS: path integral quantization; nonlinear σ model; Ward identities.

In $(2+1)$ space-time dimensions, the nonlinear σ model has been applied to study in nuclear physics (Ferrari, 2005) and condensed matter of physics (Haiperin, 1984; Laughlin, 1988a,b) for a long time. Especially, the O(3) nonlinear σ model and CP^1 nonlinear σ model have been widely discussed. They have obtained the fractional spin and fractional statistics in the O(3) nonlinear σ model by introducing Hopf term (Bowick *et al.*, 1986; Mackenzie, 1988; Wilczek and Zee, 1983; Tsurnmaru and Tsutsui, 1999), or Chern-Simons (CS) term (Banerjee, 1994; Li, 1996; Mukherjee, 1997; Panigrahi *et al.*, 1988), or MCS term (Zhang *et al.*, 2005). And they have got the similar results in the CP^1 nonlinear σ model by adding MCS term (Wang and Li, 2004; Wany and Li, 2004). They also have deduced Ward identities of the models (Jiang and Li, 2004; Wany and Li, 2004; Zhang and Li, 2004), but they have not given the description by Feynman plot. In the following we will quantize CP^1 nonlinear σ model with MCS term, deduce Ward identities of the system, and describe the Ward identities as Feynman plots

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by one-loop (Ferrari, 2005). These plots may show the action of particles in the system with fractional quantum Hall effect (FQHE).

In $(2+1)$ space-time dimensions, the Lagrangian of the CP^1 nonlinear σ model with MCS term is

$$\mathsf{L} = \frac{1}{f}(D_\mu Z_k)^*(D^\mu Z_k) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi^2}\varepsilon_{\mu\nu\lambda}A^\mu \partial^\nu A^\lambda. \tag{1}$$

Where f is coupling parameter, $k = 1, 2$. For simplicity, we define $f = 1$. And the field variable Z_k satisfies the following constraint condition

$$Z_k Z_k^* = |Z_1|^2 + |Z_2|^2 = 1; \tag{2}$$

and $D_\mu = \partial_\mu - iA_\mu$ is covariant differential. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ are CS gauge fields. In FS path-integral quantization formalism, the first constraints and the phase space generating functional of Green function are respectively (Wang and Li, 2004; Wany and Li, 2004)

$$\Lambda^0 = \pi^0 \approx 0, \tag{3}$$

$$\Lambda^1 = 2J_0 - \left(\partial_i \pi^i + \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i A_j\right) \approx 0, \tag{4}$$

$$\begin{aligned} Z[J^\alpha, K_\alpha, U^i, V^i, W^i] = \int & \mathsf{D}\phi \mathsf{D}\pi \mathsf{D}\lambda \mathsf{D}\mu \mathsf{D}\omega \\ & \times \exp\left\{i\int d^3x\left(\mathsf{L}^P_{\text{eff}} + J^\alpha \phi_\alpha + K_\alpha \pi^\alpha \right.\right. \\ & \left.\left. + U^i\lambda_i + V^i\mu_i + W^i\omega_i\right)\right\}. \end{aligned} \tag{5}$$

Here

$$\mathsf{L}^P_{\text{eff}} = \mathsf{L}^P + \mathsf{L}_m + \mathsf{L}_{gh}, \tag{6}$$

$$\mathsf{L}^P = \pi_k \dot{Z}_k + \bar{\pi}_k \dot{Z}_k^* + \pi^\mu \dot{A}_\mu - \mathsf{H}_c; \tag{7}$$

$$\mathsf{L}_m = \lambda_l \Lambda_l + \mu_n \Omega_n + \omega_i \theta_i; \tag{8}$$

$$\mathsf{L}_{gh} = 4\bar{C}(x)(Z_k(x)Z_k^*(x))^2 C(x); \tag{9}$$

$$\begin{aligned} \pi_k &= \frac{\partial \mathsf{L}}{\partial \dot{Z}_k} = (D_0 Z_k)^*; \quad \bar{\pi}^k = \frac{\partial \mathsf{L}}{\partial \dot{Z}_k^*} = D_0 Z_k; \\ \pi^i &= \frac{\partial \mathsf{L}}{\partial \dot{A}_i} = -F^{0i} + \frac{\theta}{2\pi^2}\varepsilon^{ij}A_j; \quad \pi^0 = \frac{\partial \mathsf{L}}{\partial \dot{A}_0} = 0; \end{aligned} \tag{10}$$

$(J^\alpha, K_\alpha, U^l, V^n, W^i)$ are introduced exterior sources with respect to the fields $(\phi_\alpha, \pi^\alpha, \lambda_l, \mu_n, \omega_i)$, λ_l, μ_n and ω_i are multipliers fields.

According to Dirac's conjecture (Li *et al.*, 2005), the generator of gauge transformation of the system formula (1) can be written as

$$G = \int_V d^2x[\dot{\varepsilon}(x)\Lambda^0 - \varepsilon(x)\Lambda^1] = \int_V d^2x \left\{\dot{\varepsilon}(x)\pi^0 - \varepsilon(x)\left[2J_0 - \left(\partial_i\pi^i + \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i A_j\right)\right]\right\}. \tag{11}$$

Following above the generator, we obtain the gauge transformation

$$\begin{cases} \delta Z_k = \{Z_k(x), G\} = -2iZ_k(x)\varepsilon(x), \delta Z_k^* = \{Z_k^*(x), G\} = 2iZ_k^*(x)\varepsilon(x), \\ \delta A_\mu = \{A_\mu(x), G\} = -\partial_\mu\varepsilon(x), \delta\pi^k = \{\pi^k(x), G\} = 2i\pi^k(x)\varepsilon(x), \\ \delta\bar{\pi}^k = \{\bar{\pi}^k(x), G\} = -2i\bar{\pi}^k(x)\varepsilon(x), \delta\pi^\mu = \{\pi^\mu(x), G\} = \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i\varepsilon(x). \end{cases} \tag{12}$$

Under above the transformation, the canonical Lagrangian of the system formula (1) is invariant, but the effective Lagrangian is variant which is given as

$$\left(-\nabla^2\frac{\delta}{\delta U^0} + \nabla^2\partial_0\frac{\delta}{\delta U^1} + \frac{\theta}{\pi^2}\varepsilon^{ij}\nabla^2\frac{\delta}{\delta U^1} - \partial_\mu J^\mu - 2\bar{J}^k\frac{\delta}{\delta\bar{J}^k} + 2J^k\frac{\delta}{\delta J^k} + 2\bar{K}_k\frac{\delta}{\delta\bar{K}_k} - 2K_k\frac{\delta}{\delta K_k} - \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i K_\mu\right) Z[J^\alpha, K_\alpha, U^l, V^n, W^i] = 0. \tag{13}$$

For the generating functional of Green function $Z[J^\alpha, K_\alpha, U^l, V^n, W^i]$, the generating functional of connectivity $W[J^\alpha, K_\alpha, U^l, V^n, W^i]$ and the generating functional of proper vertices $\Gamma[J^\alpha, K_\alpha, U^l, V^n, W^i]$ satisfying the following relations

$$Z[J^\alpha, K_\alpha, U^l, V^n, W^i] = \exp\{iW[J^\alpha, K_\alpha, U^l, V^n, W^i]\}; \tag{14}$$

$$\Gamma[J^\alpha, K_\alpha, U^l, V^n, W^i] = W[J^\alpha, K_\alpha, U^l, V^n, W^i] - \int d^3x(J^\alpha\phi_\alpha + K_\alpha\pi^\alpha + U^l\lambda_l + V^n\mu_n + W^i\omega_i); \tag{15}$$

$$\begin{cases} \frac{\delta W}{\delta J^\alpha} = \phi_\alpha \\ \frac{\delta\Gamma}{\delta\phi_\alpha} = -J^\alpha \end{cases}, \quad \begin{cases} \frac{\delta W}{\delta K_\alpha} = \pi^\alpha \\ \frac{\delta\Gamma}{\delta\pi^\alpha} = -K_\alpha \end{cases}, \quad \begin{cases} \frac{\delta W}{\delta V^n} = \mu_n \\ \frac{\delta\Gamma}{\delta\mu_n} - V^n \end{cases}. \tag{16}$$

and the Ward identities, the Ward identities of the generating functional of proper vertices are obtained

$$\partial_{x_1}^{\mu} \frac{\delta^3 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_2)\delta A_\mu(x_1)} + \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_{x_1}^{i} \frac{\delta^3 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_2)\delta \pi^\mu(x_1)}$$
$$= 2\delta(x_1 - x_3)\frac{\delta^2 \Gamma[0]}{\delta Z_k(x_2)\delta Z_k^*(x_1)} - 2\delta(x_1 - x_2)\frac{\delta^2 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_1)}. \quad (17)$$

We functionally differentiate formula (18) with respect to $Z_k(x)$, or $Z_k^*(x)$, or $Z_k(x)$ and $Z_k^*(x)$, and set all fields (including multiplier fields) equal zero, $A_\mu = Z_k = Z_k^* = \mu_0 = \mu_1 = \pi^k = \bar{\pi}^k = \pi^\mu = 0$. Then we obtain

$$\partial_{x_1}^{\mu} \frac{\delta \Gamma[0]}{\delta A_\mu(x_1)\delta Z_k(x_2)} + \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i \frac{\delta^2 \Gamma[0]}{\delta \pi^\mu(x_1)\delta Z_k(x_2)} + 2\delta(x_1 - x_2)\frac{\delta \Gamma[0]}{\delta Z_k(x_1)} = 0; \quad (18)$$

$$\partial_{x_1}^{\mu} \frac{\delta \Gamma[0]}{\delta A_\mu(x_1)\delta Z_k^*(x_2)} + \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_i \frac{\delta^2 \Gamma[0]}{\delta \pi^\mu(x_1)\delta Z_k^*(x_2)} + 2\delta(x_1 - x_2)\frac{\delta \Gamma[0]}{\delta Z_k^*(x_1)} = 0; \quad (19)$$

$$\partial_{x_1}^{\mu} \frac{\delta^3 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_2)\delta A_\mu(x_1)} + \frac{\theta}{2\pi^2}\varepsilon^{ij}\partial_{x_1}^{i} \frac{\delta^3 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_2)\delta \pi^\mu(x_1)}$$
$$= 2\delta(x_1 - x_3)\frac{\delta^2 \Gamma[0]}{\delta Z_k(x_2)\delta Z_k^*(x_1)} - 2\delta(x_1 - x_2)\frac{\delta^2 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_1)}. \quad (20)$$

Substituting formula (10) into formulae (19), (20) and (21), they can be simplified as

$$\partial_{x_1}^{\mu} \frac{\delta \Gamma[0]}{\delta A_\mu(x_1)\delta Z_k(x_2)} + \delta(x_1 - x_2)\frac{\delta \Gamma[0]}{\delta Z_k(x_1)} = 0; \quad (21)$$

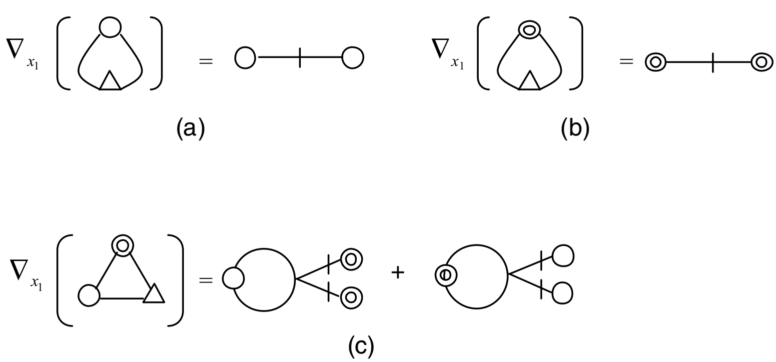

Fig. 1. Ward identities are describing by one-loop. ∇_{x_1}: three divergence, circle: $Z_k(x)$, annulus: $Z_k^*(x)$, trigon: $A_\mu(x)$, cut line: $Z_k(x)$-propagator or $Z_k^*(x)$-propagator.

$$\partial^{\mu}_{x_1} \frac{\delta \Gamma[0]}{\delta A_\mu(x_1)\delta Z_k^*(x_2)} + \delta(x_1 - x_2)\frac{\delta \Gamma[0]}{\delta Z_k^*(x_1)} = 0; \tag{22}$$

$$\begin{aligned}&\partial^{\mu}_{x_1} \frac{\delta^3 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_2)\delta A_\mu(x_1)}\\ &= \delta(x_1 - x_3)\frac{\delta^2 \Gamma[0]}{\delta Z_k(x_2)\delta Z_k^*(x_1)} - \delta(x_1 - x_2)\frac{\delta^2 \Gamma[0]}{\delta Z_k^*(x_3)\delta Z_k(x_1)}.\end{aligned} \tag{23}$$

On one-loop level, we will give the Feynman plots for two points and three points as (Fig. 1). These plots are more intuitionistic than the expressions of Ward identities. Whether the results are same for the system including Nonabel Chern–Simons term needing additive discussion.

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